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Renormalisation group flows for gauge theories in axial gauges

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Abstract: Gauge theories in axial gauges are studied using Exact Renormalisation Group flows. We introduce a background field in the infrared regulator, but not in the gauge fixing, in contrast to the usual background field gauge. We discuss the absence of spurious singularities and the finiteness of the flow. It is shown how heat-kernel methods can be used to obtain approximate solutions to the flow and the corresponding Ward identities. New expansion schemes are discussed, which are not applicable in covariant gauges. As an application, we derive the one-loop effective action for covariantly constant field strength, and the one-loop β-function for arbitrary regulator.

Keywords: Nonperturbative Effects, Renormalization Group, Renormalization Regularization and Renormalons, Confinement.

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Contents

1. Introduction

The perturbative sector of QCD is very well understood due to the weak coupling of gluons in the ultraviolet (UV) limit, known as asymptotic freedom. In the infrared (IR) region, however, the quarks and gluons are confined to hadronic states and the gauge coupling is expected to grow large. Thus the IR physics of QCD is only accessible with non-perturbative methods. The exact renormalisation group (ERG) provides such a tool [1, 2]. It is based on a regularised version of the path integral for QCD, which is solved by successively integrating-out momentum modes.

ERG flows for gauge theories have been formulated in different ways (for a review, see [3]). Within covariant gauges, ERG flows have been studied in [4, 5, 6], while general axial gauges have been employed in [7, 8]. In these approaches, gauge invariance of physical Greens functions is controlled with the help of modified Ward or Slavnov-Taylor identities [5]–[10]. A different line has been followed in [11] based on gauge invariant variables, e.g. Wilson loops. Applications of these methods to gauge theories include the physics of superconductors [12], the computation of instanton-induced effects [13], the heavy quark effective potential [14, 15], effective gluon condensation [16], Chern-Simons theory [17], monopole condensation [18], chiral gauge theories [19], supersymmetric Yang-Mills theories [20], and the derivation of the universal two-loop beta function [21].

In the present paper, we use flow equations to study Yang-Mills theories within a background field method. In contrast to the usual background field formalism [22], we use a general axial gauge, and not the covariant background field gauge. The background field enters only through the regularisation, and not via the gauge fixing. Furthermore, in axial gauges no ghost degrees of freedom are present and Gribov copies are absent. Perturbation theory in axial gauges is plagued by spurious singularities of the propagator due to an incomplete gauge fixing, which have to be regularised separately. Within an exact renormalisation group approach, and as a direct consequence of the wilsonian cutoff, these spurious singularities are absent [7]. The resulting flow equation can be used for applications even beyond the perturbative level. This formalism has been used for a study of the propagator [23], for a formulation of Callan-Symanzik flows in axial gauges [24], and for a study of Wilson loops [25, 26].

Here, we continue the analysis of [7, 8] and provide tools for the study of Yang-Mills theories within axial gauges. First we detail the discussion of the absence of spurious singularities. Then a framework for the evaluation of the path integral for covariantly constant fields is discussed. We use an auxiliary background field which allows us to define a gauge invariant effective action. The background field is introduced only in the regulator, in contrast to the usual background field formalism. This way it is guaranteed that all background field dependence vanishes in the infrared limit, where the cutoff is removed. We employ heat kernel techniques for the evaluation of the ERG flow. The heat kernel is used solely as a technical device, and not as a regularisation. The flow equation itself is by construction infra-red and ultra-violet finite and no further regularisation is required. As an explicit application, we compute the full one-loop effective action for non-abelian gauge theories. This includes the universal β-function at one loop for arbitrary regulator. We also discuss new expansions of the flow, which are not applicable for covariant gauges.

The work is organised as follows. We begin with a brief review of the wilsonian approach for gauge theories. This includes a derivation of the flow equation. We discuss the absence of spurious singularities and the finiteness of the flow. This leads to a mild restriction on the fall-off behaviour of regulators at large momenta. (Section 2). Next, we consider the implications of gauge symmetry. This includes a discussion of the Ward-Takahashi identities, the construction of a gauge-invariant effective action, and the study of the background field dependence. Explicit examples for background field dependent regulators are also given (section 3). We derive the propagator for covariantly constant

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fields, and explain how expansions in the fields and heat kernel techniques can be applied in the present framework (section 4). We compute the full one loop effective action using heat kernel techniques. We also show in some detail how the universal beta function follows for arbitrary regulator functions (section 5). We close with a discussion of the main results (section 6) and leave some more technical details to the appendices.

2. Wilsonian approach for gauge theories

In this section we review the basic ingredients and assumptions necessary for the construction of an exact renormalisation group equation for non-abelian gauge theories in general axial gauges. This part is based on earlier work [7, 8]. New material is contained in the remaining subsections, where we discuss the absence of spurious singularities and the finiteness of the flow.

2.1 Derivation of the flow

The starting point for the derivation of an exact renormalisation group equation are the classical action S_A for a Yang-Mills theory, an appropriate gauge fixing term $S_{\rm gf}$ and a regulator term ΔS_k, which introduces an infra-red cut-off scale k (momentum cut-off). This leads to a k-dependent effective action Γ_k. Its infinitesimal variation w.r.t. k is described by the flow equation, which interpolates between the gauge-fixed classical action and the quantum effective action, if ΔS_k and Γ_k satisfy certain boundary conditions at the initial scale Λ. The classical action of a non-abelian gauge theory is given by

$$S_A[A] = \frac{1}{4}\int d^4x\, F^a_{\mu\nu}(A) F^a_{\mu\nu}(A) \tag{2.1}$$

with the field strength tensor

$$F^a_{\mu\nu}(A) = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^a{}_{bc} A^b_\mu A^c_\nu \tag{2.2}$$

and the covariant derivative

$$D^{ab}_\mu(A) = \delta^{ab}\partial_\mu + g f^{acb} A^c_\mu\,, \qquad [t^b, t^c] = f_a{}^{bc} t^a\,. \tag{2.3}$$

A general axial gauge fixing is given by

$$S_{\rm gf}[A] = \frac{1}{2}\int d^4x\, n_\mu A^a_\mu \frac{1}{\xi n^2} n_\nu A^a_\nu\,. \tag{2.4}$$

The gauge fixing parameter ξ has the mass dimension -2 and may as well be operator-valued [7]. The particular examples $\xi = 0$ and $\xi p^2 = -1$ are known as the axial and the planar gauge, respectively. The axial gauge is a fixed point of the flow [7].

The scale-dependent regulator term is

$$\Delta S_k[A, \bar{A}] = \frac{1}{2}\int d^4x A^a_\mu\, R_k{}^{ab}_{\mu\nu}[\bar{A}] A^b_\nu\,. \tag{2.5}$$

It is quadratic in the gauge field and leads to a modification of the propagator. We have introduced a background field $\bar{A}$ in the regulator function. Both the classical action and the gauge fixing depend only on A. The background field serves as an auxiliary field which can be interpreted as an index for a family of different regulators $R_{k,\bar{A}}$. Its use will become clear below.

The scale dependent Schwinger functional $W_k[J,\bar{A}]$, given by

$$\exp W_k[J,\bar{A}] = \int \mathcal{D}A \exp\left\{-S_k[A,\bar{A}] + \int d^4x\, A_\mu^a J_\mu^a\right\}, \tag{2.6}$$

where

$$S_k[A,\bar{A}] = S_A[A] + S_{\rm gf}[A] + \Delta S_k[A,\bar{A}]\,. \tag{2.7}$$

We introduce the scale dependent effective action $\Gamma_k[A,\bar{A}]$ as the Legendre transform of (2.6)

$$\Gamma_k[A,\bar{A}] = \int d^4x J_\mu^a A_\mu^a - W_k[J,\bar{A}] - \Delta S_k[A,\bar{A}]\,, \qquad A_\mu^a = \frac{\delta W_k[J,\bar{A}]}{\delta J_\mu^a}\,. \tag{2.8}$$

For later convenience, we have subtracted ΔS_k from the Legendre transform of W_k. Thus $\Gamma_k[A,\bar{A}]$ is given by the integro-differential equation

$$\exp -\Gamma_k[A,\bar{A}] = \int \mathcal{D}a\, \exp\left\{-S_A[a] - S_{\rm gf}[a] - \Delta S_k[a-A,\bar{A}] + \frac{\delta}{\delta A}\Gamma_k[A,\bar{A}](a-A)\right\}. \tag{2.9}$$

The corresponding flow equation for the effective action

$$\partial_t \Gamma_k[A,\bar{A}] = \frac{1}{2}{\rm Tr}\left\{G_k[A,\bar{A}]\,\partial_t R_k[\bar{A}]\right\} \tag{2.10}$$

follows from (2.9) by using $\langle a - A\rangle = 0$. The trace sums over all momenta and indices, $t = \ln k$. G_k is the full propagator of the field A, whereas $\bar{A}$ is not propagating. Its inverse is given by

$$\left(G_k[A,\bar{A}]\right)^{-1\ ab}_{\ \ \mu\nu}(x,x') = \frac{\delta^2\Gamma_k[A,\bar{A}]}{\delta A_a^\mu(x)\delta A_b^\nu(x')} + R_k[\bar{A}]^{\ ab}_{\mu\nu}(x,x')\,. \tag{2.11}$$

There are no ghost terms present in (2.10) due to the axial gauge fixing. For the regulator R_k we require the following properties at $\bar{A}=0$.

$$\lim_{p^2/k^2\to\infty} p^2 R_k = 0\,, \qquad \lim_{p^2/k^2\to 0} R_k \sim p^2\left(\frac{k^2}{p^2}\right)^\gamma, \tag{2.12}$$

where p^2 is plain momentum squared. Regulators with $\gamma = 1$ have a mass-like infra-red behaviour with $R_k(0)\sim k^2$. The example in (3.21) has $\gamma = 1$. In turn, regulator with $\gamma > 1$ diverge for small momenta. The latter condition in (2.12) implies that R_k introduces an IR regularisation into the theory. The first condition in (2.12) ensures the UV finiteness of the flow in case that $G_k \propto p^{-2}$ for large p^2. For covariant gauges this is guaranteed. Within axial gauges, additional care is necessary because of the presence of spurious singularities. It is seen by inspection of (2.9) and (2.12) that the saddle-point approximation about A becomes exact for $k\to\infty$. Here, Γ_k approaches the classical action. For $k \to 0$, in turn, the cut-off term disappears and we end up with the full quantum action. Hence, we confirmed that the functional Γ_k indeed interpolates between the gauge-fixed classical and the full quantum effective action:

$$\lim_{k\to\infty} \Gamma_k[A,\bar{A}] \equiv S[A] + S_{\rm gf}[A]\,, \tag{2.13a}$$

$$\lim_{k\to 0} \Gamma_k[A,\bar{A}] \equiv \Gamma[A]\,. \tag{2.13b}$$

Notice that both limits are independent of $\bar{A}$ supporting the interpretation of $\bar{A}$ as an index for a class of flows. It is worth emphasising that both the infrared and ultraviolet finiteness of (2.10) are ensured by the conditions (2.12) on R_k.

2.2 Absence of spurious singularities

The flow equation (2.10) with a choice for the initial effective action Γ_Λ at the initial scale Λ serves upon integration as a definition of the full effective action $\Gamma = \Gamma_{k=0}$. It remains to be shown that (2.10) is finite for all k thus leading to a finite Γ. In particular this concerns the spurious singularities present in perturbation theory: the propagator $P_{\mu\nu}$ related to $S = S_A + S_{\rm gf}$ is

$$P_{\mu\nu} = \frac{\delta_{\mu\nu}}{p^2} + \frac{n^2(1+\xi p^2)}{(np)^2}\frac{p_\mu p_\nu}{p^2} - \frac{1}{p^2}\frac{(n_\mu p_\nu + n_\nu p_\mu)}{np}\,. \tag{2.14}$$

It displays the usual IR poles proportional to $1/p^2$. We observe additional divergences for momenta orthogonal to n_μ. These poles appear explicitly up to second order in $1/(np)$ and can even be of higher order for certain (np)-dependent choices of ξ. For the planar gauge $\xi p^2 = -1$, the spurious divergences appear only up to first order.

This artifact makes the application of perturbative techniques very cumbersome as an additional regularisation for these spurious singularities has to be introduced. We argued in [7] that these spurious singularities are missing in the flow equation. Here, we further the discussion, also providing some information about the intricate limit where the cut-off is removed. First of all we derive a bound on the flow (2.10). Then, we argue that this bound results in weak constraints on the decay behaviour of the regulator function r for large momenta. This is sufficient for providing a well-defined RG flow.

We start with an analysis of the momentum dependence of the propagator in the presence of the regulator. To that end we set the background field to zero, $\bar{A} = 0$, and specify the regulator as

$$R^{ab}_{k,\mu\nu}(p) = \delta^{ab}\left[r(p^2)p^2\delta_{\mu\nu} - \tilde{r}(p^2)p_\mu p_\nu\right]. \tag{2.15}$$

The IR/UV limits of $r, \tilde{r}$ can be read-off from (2.12). In (2.15) we did not introduce terms with tensor structure $(n_\mu p_\nu + n_\nu p_\mu)$ and $n_\mu n_\nu$. For the present purpose, the discussion of spurious singularities, the choice (2.15) suffices. Indeed, even $\tilde{r}$ plays no rôle for the absence of spurious singularities in the flow equation approach. The only important term for the discussion of spurious singularities is that proportional to the term $p^2\delta_{\mu\nu}\delta^{ab}$. It is this term, proportional to the identity operator, that guarantees the suppression of

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all momentum modes for large cut-off. The other tensor structures are proportional to projection operators and cannot lead to a suppression of all modes. With a regulator obeying (2.15) the propagator takes the form

$$P_{k,\mu\nu} = a_1\frac{\delta_{\mu\nu}}{p^2} + a_2\frac{p_\mu p_\nu}{p^4} + a_3\frac{n_\mu p_\nu + n_\nu p_\mu}{p^2(np)} + a_4\frac{n_\mu n_\nu}{n^2p^2}\,, \tag{2.16}$$

with the dimensionless coefficients

$$a_1 = \frac{1}{(1+r)}\,, \tag{2.17a}$$

$$a_2 = \frac{(1+\tilde{r})(1+\xi p^2(1+r))}{z}\,, \tag{2.17b}$$

$$a_3 = -\frac{(1+\tilde{r})s^2}{z}\,, \tag{2.17c}$$

$$a_4 = -\frac{(r-\tilde{r})}{z}\,, \tag{2.17d}$$

and

$$s^2 = \frac{(np)^2}{(n^2p^2)}\,, \tag{2.17e}$$

$$z = (1+r)[(1+\tilde{r})s^2 + (r-\tilde{r})(1+p^2\xi(1+r))]\,. \tag{2.17f}$$

Now we evaluate the different limits in p^2 and k important for the approach. To keep things simple we restrict ourselves to the case $\tilde{r}=0$ and a regulator r leading to a mass-like IR behaviour: $\lim_{p^2/k^2\to 0} r(p^2) = k^2/p^2$. For this choice we deduce from (2.16) and (2.12) that $P_{k,\mu\nu}$ has the limits

$$\lim_{p^2/k^2\to\infty} P_{k,\mu\nu} = P_{\mu\nu}\,, \qquad \lim_{p^2/k^2\to 0} P_{k,\mu\nu} = \frac{1}{k^2}\left(\delta_{\mu\nu} + \frac{n_\mu n_\nu}{n^2}\frac{1}{1+\xi k^2}\right), \tag{2.18}$$

with $P_{\mu\nu}$ defined in (2.14). By construction, the propagator (2.16) is IR finite for any $k>0$. Now, the important observation is the following: in contrast to the perturbative propagator $P_{\mu\nu}$, the limit of $P_{k,\mu\nu}$ for $np\to 0$ is finite. This holds true even for an arbitrary choice of $\xi(p,n)$ and leads to

$$P_{k,\mu\nu} = \frac{1}{1+r}\frac{\delta_{\mu\nu}}{p^2} + \frac{1}{(1+r)r}\frac{p_\mu p_\nu}{p^4} - \frac{1}{(1+r)(1+p^2\xi(1+r))}\frac{n_\mu n_\nu}{n^2p^2}\,. \tag{2.19}$$

Thus (2.19) is well-behaved and finite for all momenta p. The plain spurious divergences are already absent as soon as the infra-red behaviour of the propagator is under control. This holds for R with the most general tensor structure as long as it obeys the limits (2.12). For example, it is easily extended to non-zero $\tilde{r}$ as long as the regulators r and $\tilde{r}$ have not been chosen to be identical. Already in the infrared region $\tilde{r}$ has to be smaller than r in order to have a suppression of longitudinal modes at all. So we discard the option of identical r and $\tilde{r}$.

Still, for $np=0$ and large momenta squared $y=p^2$ the regulator tends to zero and the second term in (2.19) diverges in the limit $y\to\infty$ proportional to $y^{-1}(r-\tilde{r})^{-1} > y^{d/2-1}$, following from (2.12). Hence, even though the term only diverges for $y\to\infty$, a more careful analysis is needed for proving the finiteness of the flow equation. We emphasise that the remaining problem is the integration over *large momenta* in the flow equation and *not* an IR problem at vanishing momentum. Thus, by showing that this problem is absent in the flow equation for all k it cannot reappear at $k = 0$. Indeed, we shall see that finiteness of the flow for all k implies a stronger decay of the regulator for large momenta as in (2.12). In turn, one may expect problems for regulators with weaker decay.

2.3 Finiteness of the RG flow

Here, finiteness of the flow equation is proven by deriving an upper bound for the flow following a bootstrap approach. The derivation of the flow equation is based on the existence of a finite renormalised Schwinger functional for the full theory [21]. In the present context this only implies the *existence* of a renormalisation procedure for axial gauges, the form of which is then determined by integrating the flow. An explicit systematic constructive renormalisation procedure is not required. The latter is a problem in perturbative field theory: no renormalisation procedure is known, which can be proven to be valid to all orders of perturbation theory.

In the present approach, spurious singularities could spoil the finiteness of (2.10) due to infinities arising from the integration of the large momentum domain. For the derivation of a bound we can safely assume, that for all k and large momenta p^2 the full propagator $\Gamma_k^{(2)}$ is dominated by its classical part (possibly with some multiplicative renormalisation constants). Hence for large momenta we can estimate $\Gamma_k^{(2)}(S^{(2)} + S_{\rm gf}^{(2)})^{-1} < C[A, \bar A]$ with $C[A, \bar A] > 0$. Consequently the field independent part of the flow provides a bound on the full flow. The only terms that could produce divergences are related to the terms in (2.16) proportional to a_2 and a_3, the source for divergences being z^{-1}. The coefficient a_4 of the last term in (2.16) also contains z^{-1} but also an additional factor r. Hence the limit $np \to 0$ can be safely done in the term a_4.

We do not go into the details of the computation. A more detailed derivation and discussion is given elsewhere. We quote the result for $\tilde r = 0$. Upon integrating the angular s-part of the momentum integration we get an estimate from the part of $\mathrm{Tr}\, P_k\, \partial_t R_k$ with the slowest decay for $y \to \infty$

$$\text{bound} \propto \left| \int_a^\infty dy\, y^2 \frac{\sqrt{1 + y\xi}}{1 + a\xi}\, \frac{r'(y)}{\sqrt{r(y)}} \right|, \tag{2.20}$$

where the square root terms stem from an integration $\int_{-1}^1 ds/[s^2 + (1 + \xi y\,)r(y)]$. Since the potential problem only occurs from an integration over large momenta squared $y = p^2$, we have restricted the y-integral to $y \geq a$ where a is at our disposal. It can be chosen the same for all k. This ensures that the limit $k \to 0$ can be taken smoothly. The bound (2.20) stems from the second term in (2.16) proportional to a_2. Eq. (2.20) is finite for regulators r that decay faster than y^{-5}. Without spurious singularities, r has to decay stronger than y^{-2}, see (2.12). Hence we have a mild additional constraint due to the fact that the full propagator G_k does not introduce an additional suppression. Typically, the regulator is chosen to decay exponentially for large momenta. Similar finite integrals as in (2.20) also occur in field dependent terms in the flow, as we shall see later in section 5.

This analysis shows the finiteness of the flow (2.10) and supports the claim that the flow equation provides a consistent quantisation procedure for gauge theories in axial gauges. The bound also marks the use of Callan-Symanzik (CS) type flows ($R_k \propto k^2$ and $r(y) \propto y^{-1}$) as questionable in axial gauges. It has been already mentioned in [3] that such a choice requires an additional renormalisation. The presence of contributions from all momenta at every flow step makes the limit $k \to 0$ an extremely subtle one. This limit is very sensitive to a proper fine-tuning. In axial gauges, this problem for CS flows gets even worse due to the spurious singularities. We know that a consistent renormalisation procedure in the axial gauge is certainly non-trivial. For CS-type flows, one is back to the original problem of spurious singularities in perturbation theory, but with a more difficult propagator and additional renormalisation problems. A recent calculation of perturbative corrections to the Wilson loop has indeed shown that formulations in axial gauge with a mass term for the gauge field meet problems [25, 26]. The massless limit of this observable did not coincide with the well-known result. In turn, for regulators which decay faster than $r(y) \sim y^{-5}$, the problem is cured.

3. Symmetries

In this section, the issue of gauge invariance of physical Greens functions, controlled by modified Ward-Takahashi identities, is studied. We discuss the role of background fields, which, in contrast to the usual background field method [22, 8], will only be introduced for the wilsonian regulator term. The Ward-Takahashi identities for the quantum and the background field are derived. We define a gauge-invariant effective action as it follows from the present formalism, and discuss its background field dependence. Finally, we discuss the background field dependent regularisation.

3.1 Modified Ward-Takahashi identities

We now address the issue of gauge invariance for physical Greens functions. The problem to face is that the presence of a regulator term quadratic in the gauge fields is, a priori, in conflict with the requirements of a (non-linear) gauge symmetry. This question has been addressed earlier for wilsonian flows within covariant gauges [4, 5, 6, 7, 9]. The resolution to the problem is that modified Ward-Takahashi identities (as opposed to the usual ones) control the flow such that physical Greens functions, obtained from Γ_k at $k = 0$, satisfy the usual Ward-Takahashi identities.

The same line of reasoning applies in the present case even though in the presence of the background field $\bar{A}$ some refinement is required [8]. In this particular point it is quite similar to the symmetry properties of the full background field formalism as discussed in [10]. The background field makes it necessary to deal with two kinds of modified Ward-Takahashi identities. The first one is related to the requirement of gauge invariance for physical Green functions, and is known as modified Ward identity (mWI). The second one has to do with the presence of a background field $\bar{A}$ in the regulator term R_k, and will be denoted as the background field Ward-Takahashi identity (bWI).

To simplify the following expressions let us introduce the abbreviation $\boldsymbol{\delta}_\omega$ and $\bar{\boldsymbol{\delta}}_\omega$ for the generator of gauge transformations on the fields A and $\bar{A}$ respectively:

$$\boldsymbol{\delta}_\omega A = D(A)\omega \qquad \boldsymbol{\delta}_\omega \bar{A} = 0 \tag{3.1a}$$

$$\bar{\boldsymbol{\delta}}_\omega A = 0 \qquad \bar{\boldsymbol{\delta}}_\omega \bar{A} = D(\bar{A})\omega\,. \tag{3.1b}$$

The action of the gauge transformations $\boldsymbol{\delta}_\omega$ and $\bar{\boldsymbol{\delta}}_\omega$ on the effective action Γ_k can be computed straightforwardly. It is convenient to define

$$\mathcal{W}_k[A,\bar{A};\omega] \equiv \boldsymbol{\delta}_\omega\Gamma_k[A,\bar{A}] - \mathrm{Tr}\,(n_\mu\partial_\mu\omega)\,\frac{1}{n^2\xi}n_\nu A_\nu + \frac{1}{2}\mathrm{Tr}\,\omega\left[G_k[A,\bar{A}],R_k[\bar{A}]\right] \tag{3.2a}$$

$$\bar{\mathcal{W}}_k[A,\bar{A};\omega] \equiv \bar{\boldsymbol{\delta}}_\omega\Gamma_k[A,\bar{A}] - \frac{1}{2}\mathrm{Tr}\,\omega\left[G_k[A,\bar{A}],R_k[\bar{A}]\right]. \tag{3.2b}$$

In terms of (3.2), the behaviour of $\Gamma_k[A,\bar{A}]$ under the transformations $\boldsymbol{\delta}_\omega$ and $\bar{\boldsymbol{\delta}}_\omega$, respectively, is given by

$$\mathcal{W}_k[A,\bar{A};\omega] = 0 \tag{3.3a}$$

$$\bar{\mathcal{W}}_k[A,\bar{A};\omega] = 0 \tag{3.3b}$$

Eq. (3.3b) is valid for regulators R_k that transform as tensors under $\bar{\boldsymbol{\delta}}_\omega$,

$$\bar{\boldsymbol{\delta}}_\omega R_k[\bar{A}] = \left[R_k[\bar{A}],\omega\right]. \tag{3.4}$$

Eq. (3.3a) is referred to as the modified Ward-Takahashi identity, and (3.3b) as the background field Ward-Takahashi identity.

Let us show that (3.3) is consistent with the basic flow equation (2.10). With consistency, we mean the following. Assume, that a functional Γ_k is given at some scale k which is a solution to both the mWI and the bWI. We then perform a small integration step from k to $k' = k - \Delta k$, using the flow equation, and ask whether the functional $\Gamma_{k'}$ again fulfils the required Ward identities (3.3). That this is indeed the case is encoded in the following flow equations for (3.3), namely

$$\partial_t\mathcal{W}_k[A,\bar{A};\omega] = -\frac{1}{2}\mathrm{Tr}\left(G_k\frac{\partial R_k}{\partial t}G_k\frac{\delta}{\delta A}\otimes\frac{\delta}{\delta A}\right)\mathcal{W}_k[A,\bar{A};\omega] \tag{3.5a}$$

$$\partial_t\bar{\mathcal{W}}_k[A,\bar{A};\omega] = \frac{1}{2}\mathrm{Tr}\left(G_k\frac{\partial R_k}{\partial t}G_k\frac{\delta}{\delta A}\otimes\frac{\delta}{\delta A}\right)\bar{\mathcal{W}}_k[A,\bar{A};\omega]\,, \tag{3.5b}$$

where $\left(\frac{\delta}{\delta A}\otimes\frac{\delta}{\delta A}\right)^{ab}_{\mu\nu}(x,y) = \frac{\delta}{\delta A^\mu_a(x)}\frac{\delta}{\delta A^\nu_b(y)}$. (3.5) states that the flow of mWI is zero if the mWI is satisfied for the initial scale. The required consistency follows from the fact that the flow is proportional to the mWI itself (3.5a), which guarantees that (3.3a) is a fixed point of (3.5a). The same follows for the bWI by using (3.5b). There is no fine-tuning involved in lifting a solution to (3.3a) to a solution to (3.3b). It also straightforwardly follows from (3.5a) and (3.5b).

We close with a brief comment on the use of mass term regulators. Such a regulator corresponds simply to $R_k = k^2$ and leads to a Callan-Symanzik flow. The regulator is momentum-independent, which implies that the loop term in (3.2a) vanishes identically. Hence one concludes that the modified Ward identity reduces to the usual one for all scales k. This happens only for an axial gauge fixing [7].

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3.2 Gauge invariant effective action

Returning to our main line of reasoning and taking advantage of the results obtained in the previous section, we define a gauge invariant effective action only dependent on A by identifying $\bar{A} = A$. It is obtained for a particular choice of the background field, and provides the starting point for our formalism.

It is a straightforward consequence of the mWI (3.3a) and the bWI (3.3b) that the effective action $\Gamma_k[A, \bar{A}]$ is gauge invariant — up to the gauge fixing term — under the combined transformation

$$(\boldsymbol{\delta}_\omega + \bar{\boldsymbol{\delta}}_\omega)\Gamma_k[A, \bar{A}] = \mathrm{Tr}\, n_\mu(\partial_\mu\omega)\frac{1}{n^2\xi}n_\nu A_\nu\,. \tag{3.6}$$

We define the effective action $\hat{\Gamma}_k[A]$ as

$$\hat{\Gamma}_k[A] = \Gamma_k[A, \bar{A} = A]\,. \tag{3.7}$$

The action $\hat{\Gamma}_k[A]$ is gauge invariant up to the gauge fixing term, to wit

$$\boldsymbol{\delta}_\omega\hat{\Gamma}_k[A] = \mathrm{Tr}\left\{n_\mu(\partial_\mu\omega)\frac{1}{n^2\xi}n_\nu A_\nu\right\}. \tag{3.8}$$

This follows from (3.6). Because of (2.13b), the effective action $\hat{\Gamma}_{k=0}[A]$ is the full effective action. The flow equation for $\hat{\Gamma}_k[A]$ can be read off from the basic flow equation (2.10),

$$\partial_t\hat{\Gamma}_k[A] = \frac{1}{2}\mathrm{Tr}\left\{G_k[A, A]\,\partial_t R_k[A]\right\}, \tag{3.9}$$

Notice that the right-hand side of (3.9) is not a functional of $\hat{\Gamma}_k[A]$. The flow depends on the full propagator $G_k[A, A]$, which is the propagator of A in the background of $\bar{A}$ taken at $\bar{A} = A$. Thus for the flow of $\hat{\Gamma}_k[A]$ one needs to know the flow (of a subset) of vertices of $\delta^2\Gamma_k[A, \bar{A}]/(\delta A)^2$ at $\bar{A} = A$. Still, approximations, where this difference is neglected are of some interest [27].

We argue that (3.8) has far reaching consequences for the renormalisation procedure of $\hat{\Gamma}_k[A]$ as is well-known for axial gauges and the background field formalism. $\Gamma_k[A]$ is gauge invariant up to the breaking due to the gauge fixing term. We define its gauge invariant part as

$$\Gamma_{k,\mathrm{inv}}[A] = \Gamma_k[A] - S_{\mathrm{gf}}[A] \tag{3.10a}$$

$$\boldsymbol{\delta}_\omega\Gamma_{k,\mathrm{inv}}[A] = 0\,. \tag{3.10b}$$

Eq. (3.10) implies that the combination gA is invariant under renormalisation, $\partial_t(gA) = 0$. If one considers wave function renormalisation and coupling constant renormalisation for A and g respectively

$$A \to Z_F^{1/2}A \tag{3.11a}$$

$$g \to Z_g g \tag{3.11b}$$

we conclude that

$$Z_g = Z_F^{-1/2}\,. \tag{3.12}$$

3.3 Background field dependence

By construction, the effective action $\Gamma_k[A,\bar{A}]$ at some finite scale $k\neq 0$ will depend on the background field $\bar{A}$. This dependence disappears for $k=0$. The effective action $\hat{\Gamma}_k[A]$ is the simpler object to deal with as it is gauge invariant and only depends on one field. As we have already mentioned below (3.9), its flow depends on the the propagator $\delta_A^2\Gamma_k[A,\bar{A}]$ at $A=\bar{A}$. Eventually we are interested in approximations where we substitute this propagator by $\delta_A^2\hat{\Gamma}_k$. The validity of such an approximation has to be controlled by an equation for the background field dependence of $\Gamma_k[A,\bar{A}]$. The flow of the background field dependence of $\Gamma_k[A,\bar{A}]$ can be derived in two ways. $\delta_{\bar{A}}\partial_t\Gamma_k$ can be derived from the flow equation (2.10),

$$\frac{\delta}{\delta\bar{A}}\partial_t\Gamma_k[A,\bar{A}]=\frac{1}{2}\frac{\delta}{\delta\bar{A}}\,\mathrm{Tr}\left\{G_k[A,\bar{A}]\partial_t R_k[\bar{A}]\right\}. \tag{3.13}$$

The flow $\partial_t\delta_{\bar{A}}\Gamma_k$ follows the observation that the only background field dependence of Γ_k originates in the regulator. Thus, $\delta_{\bar{A}}\Gamma_k$ is derived along the same lines as the flow itself and we get

$$\partial_t\frac{\delta}{\delta\bar{A}}\Gamma_k[A,\bar{A}]=\frac{1}{2}\mathrm{Tr}\,\partial_t\left\{G_k[A,\bar{A}]\frac{\delta}{\delta\bar{A}}R_k[\bar{A}]\right\}, \tag{3.14}$$

which turns out to be important also for the derivation of the universal one loop β-function in section 5.2. The difference of (3.13) and (3.14) has to vanish

$$\left[\frac{\delta}{\delta\bar{A}}\,,\,\partial_t\right]\Gamma_k[A,\bar{A}]=0\,. \tag{3.15}$$

Eq. (3.15) combines the flow of the intrinsic $\bar{A}$-dependence of $\Gamma_k[A,\bar{A}]$ (3.14) with the $\bar{A}$-dependence of the flow equation itself (3.13). It provides a check for the validity of a given approximation. Using the right hand sides of (3.13) and (3.14) the consistency condition (3.15) can be turned into

$$\mathrm{Tr}\left\{G_k\,\frac{\delta\Gamma_k^{(2)}}{\delta\bar{A}}G_k\,\partial_t R_k\right\}=\mathrm{Tr}\left\{G_k\,\frac{\delta R_k}{\delta\bar{A}}\,G_k\partial_t\Gamma_k^{(2)}\right\}, \tag{3.16}$$

where

$$\Gamma_k^{(2)}[A,\bar{A}]^{\ ab}_{\mu\nu}(x,x')=\frac{\delta^2\Gamma_k[A,\bar{A}]}{\delta A_a^\mu(x)\delta A_b^\nu(x')}\,. \tag{3.17}$$

With (3.16), we control the approximation

$$\left.\frac{\delta^2\Gamma_k[A,\bar{A}]}{\delta A\,\delta A}\right|_{\bar{A}=A}=\frac{\delta^2\hat{\Gamma}_k[A]}{\delta A\,\delta A}+\text{sub-leading terms} \tag{3.18}$$

For this approximation the flow (3.9) is closed and can be calculated without the knowledge of $\Gamma_k^{(2)}$, but with $\hat{\Gamma}_k^{(2)}$. Amongst others, the approximation (3.18) is implicitly made within proper-time flows, where the use of heat-kernel methods is even more natural [28]. This is discussed in [27, 29]. Let us finally comment on the domain of validity for the approximation (3.18). In the infrared $k\to 0$, the dependence of the effective action $\Gamma_k[A,\bar{A}]$ on the background field $\bar{A}$ becomes irrelevant, because the regulator $R_k[\bar{A}]$ tends to zero. Therefore we can expect that (3.18) is reliable in the infrared, which is the region of interest.

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3.4 Regulators

We have seen that the symmetries of the effective action Γ_k and the flow crucially depend on the properties of $R_k[\bar{A}]$, in particular the construction of a gauge invariant effective action. The regulator has to transform as a tensor under gauge transformations of $\bar{A}$, (3.4). Here we specify a general class of regulators which has this property and is well-suited for practical applications. As already argued in section 2.2, the infrared regularisation is provided by r, whereas $\tilde{r} \neq 0$ only gives different weights to the longitudinal degrees of freedom, see (2.15). In the following we set $\tilde{r} \equiv 0$. We choose

$$R_k[\bar{A}] = \bar{D}_T \, r(\bar{D}_T) \tag{3.19}$$

with the yet unspecified function r. We introduced D_T, the Laplace operator for spin 1,

$$D^{ab}_{T,\mu\nu}(A) := -(D_\rho D_\rho)^{ab}(A)\delta_{\mu\nu} - 2gF^{ab}_{\mu\nu}(A) \tag{3.20}$$

and $\bar{D}_T = D_T(\bar{A})$. For vanishing background field the Laplacean D_T reduces to the free Laplacean $D_T(0) = p^2$. In this case we have $R_k = p^2 \, r(p^2)$. Written in terms of some general Laplace operator $P^2(\bar{A})$, a typical example for the regulator functions $R_k(P^2)$ and $r(P^2)$ is

$$R_k(P^2) = \frac{P^2}{\exp P^2/k^2 - 1}\,, \qquad r(P^2) = \frac{1}{\exp P^2/k^2 - 1} \tag{3.21}$$

which meets the general properties as described in (2.12). Eq. (3.21) is an example for a regulator with a mass-like IR behaviour, $\gamma = 1$. More generally the IR/UV conditions for R_k in (2.12) translates into

$$\lim_{k^2/p^2 \to 0} \left(\frac{p^2}{k^2}\right)^2 r = 0\,, \qquad \lim_{P^2 \to 0} r \sim \left(\frac{k^2}{p^2}\right)^\gamma \tag{3.22}$$

for the function r.

4. Analytic methods

In this section we develop analytical methods to study flow equations for gauge theories in general axial gauges. The flow equation is a one-loop equation which makes it possible to use heat kernel techniques for its solution. The main obstacles, technically speaking, are the constraint imposed by the modified Ward identity and the necessity to come up with a closed form for the full propagator. We first derive such an expression for the case of covariantly constant fields within general axial gauges. In addition a generic expansion procedure in powers of the fields is discussed. Finally, we give the basic heat kernels to be employed in the next section.

4.1 Propagator for covariantly constant fields

We derive an explicit expression for the full propagator for specific field configurations. This is a prerequisite for the evaluation of the flow equation (2.10). To that end we restrict ourselves to field configurations with covariantly constant field strength (see e.g. [30]), namely $D_\mu F_{\nu\rho} = 0$. This is a common procedure within the algebraic heat kernel approach. We also use the existence of the additional Lorentz vector to demand $n_\mu A^\mu = n_\mu F_{\mu\nu} = 0$. That this can be achieved is proven by the explicit example of $n_\mu = \delta_{\mu 0}$ and $(A_\mu) = (A_0 = 0, A_i(\vec{x}))$. These constraints lead to

$$[D_\mu, F_{\nu\rho}] = 0\,, \tag{4.1a}$$

$$n_\mu A_\mu = 0\,, \tag{4.1b}$$

$$n_\mu F_{\mu\nu} = 0\,. \tag{4.1c}$$

To keep finiteness of the action of such configurations we have to go to a theory on a finite volume. However, the volume dependence will drop out in the final expressions and we smoothly can take the limit of infinite volume. For the configurations satisfying (4.1) we derive the following properties

$$[D^2, D_\mu] = -2gF_{\mu\rho}D_\rho\,, \tag{4.2a}$$

$$D_{T,\mu\rho}D_\rho = -D_\mu D^2\,, \tag{4.2b}$$

$$[n_\rho D_\rho, D_\mu] = 0\,. \tag{4.2c}$$

Defining the projectors P_n and P_D with

$$P_{n,\mu\nu} = \frac{n_\mu n_\nu}{n^2}\,, \tag{4.3a}$$

$$P_{D,\mu\nu} = D_\mu \frac{1}{D^2} D_\nu \tag{4.3b}$$

we establish that

$$P_D D_T = -P_D D^2 P_D, \qquad P_n D_T = -P_n D^2 \tag{4.4}$$

holds true. After these preliminary considerations we consider the gauge-fixed classical action given in (2.1).We need the propagator on tree level to obtain the traces on one-loop level. The initial action reads

$$\Gamma_\Lambda[A] = S_A + S_{\rm gf}\,. \tag{4.5}$$

From (4.5) we derive the full inverse propagator as

$$\Gamma^{(2)ab}_{k,\mu\nu}[A,A] = \left(D^{ab}_{T,\mu\nu} + (D_\mu D_\nu)^{ab} + \frac{1}{\xi n^2} n_\mu n_\nu \delta^{ab}\right) + O(g^2; D_T, D_\mu D_\nu)\,. \tag{4.6}$$

The inverse propagator (4.6) is an operator in the adjoint representation of the gauge group. We now turn to the computation of the propagator (2.11) for covariantly constant fields. Using (4.6), (4.1) and (4.2), we find

$$G_k[A,A]^{\,ab}_{\,\mu\nu} = -\left(\left(\frac{a_1}{D_T}\right)_{\mu\nu} + D_\mu \frac{a_2}{D^4} D_\nu + n_\mu \frac{a_3}{D^2(nD)} D_\nu + D_\mu \frac{a_3}{D^2(nD)} n_\nu + \frac{n_\mu a_4 n_\nu}{n^2 D^2}\right), \tag{4.7}$$

with the dimensionless coefficient functions

$$a_1 = \frac{1}{1+r_T}\,, \tag{4.8a}$$

$$a_2 = \frac{1-\xi D^2(1+r_D)}{(1+r_D)}\left(s^2 + r_D[1-D^2\xi(1+r_D)]\right)^{-1}\,, \tag{4.8b}$$

$$a_3 = -\frac{s^2}{(1+r_D)}\left(s^2 + r_D[1-D^2\xi(1+r_D)]\right)^{-1}\,, \tag{4.8c}$$

$$a_4 = -\frac{r_D}{(1+r_D)}\left(s^2 + r_D[1-D^2\xi(1+r_D)]\right)^{-1}\,. \tag{4.8d}$$

Notice that a_1 is a function of D_T while a_2, a_3 and a_4 are functions of both D^2 and $(nD)^2$. We also introduced the convenient short-hands

$$r_T \equiv r_k(D_T)\,, \qquad r_D \equiv r_k(-D^2)\,, \qquad s^2 \equiv \frac{(nD)^2}{(n^2 D^2)}\,. \tag{4.9}$$

The regulator function, as introduced in (3.19), depends on D_T. The dependence on D^2, as apparent in the terms a_2, a_3 and a_4, comes into game due to the conditions (4.1) and (4.2). They imply

$$r_k(D_T)D_\mu = D_\mu r_k(-D^2)\,, \qquad r_k(D_T)n_\mu = n_\mu r_k(-D^2)\,, \tag{4.10}$$

which can be shown term by term for a Taylor expansion of r_k about vanishing argument. For vanishing field $A=0$ the propagator (4.7) reduces to the one already discussed in [7]. There, it has been shown that the regularised propagator (4.7) (for $r \neq 0$) is not plagued by the spurious propagator singularities as encountered within standard perturbation theory, and in the absence of a regulator term ($r=0$). For the axial gauge limit $\xi = 0$ the expression (4.7) simplifies considerably. With (4.6) and (4.9) we get

$$\begin{aligned} G_{k,\mu\nu}[A] =& \left(\frac{1}{D_T(1+r_T)}\right)_{\mu\nu} - D_\mu \frac{1}{D^4(1+r_D)(s^2+r_D)} D_\nu + \frac{n_\mu}{n^2}\frac{nD}{D^4(1+r_D)(s^2+r_D)} D_\nu + \\ &+ D_\mu \frac{nD}{D^4(1+r_D)(s^2+r_D)}\frac{n_\nu}{n^2} + \frac{r_D}{D^2(1+r_D)(s^2+r_D)} P_{n,\mu\nu}\,. \end{aligned} \tag{4.11}$$

The propagators (4.7) and (4.11) are at the basis for the following computations. Notice that this analysis straightforwardly extends to approximations for $\Gamma_k[A,\bar{A}]$ beyond the one-loop level. Indeed, it applies for any $\Gamma_k[A,\bar{A}]$ such that $\Gamma^{(2)}_{k,\mu\nu}[A,A]$ is of the form

$$\Gamma^{(2)}_{k,\mu\nu}[A,A] = f_k^{D_T}\, D_{T\ \mu\nu} + D_\mu\, f_k^{DD}\, D_\nu + n_\mu \frac{f_k^{nD}}{nD} D_\nu + D_\mu \frac{f_k^{nD}}{nD} n_\nu + n_\mu\, f_k^{nn}\, n_\nu\,. \tag{4.12}$$

Here, the scale-dependent functions $f_k^{D_T}$ and f_k^{DD} can depend on D_T, D^2 and nD. In turn, the functions f_k^{nD} and f_k^{nn} can depend only on D^2 and nD. An explicit analytical expression for the full propagator, similar to (4.7), follows from (4.12). Such approximations take the full (covariant) momentum dependence of the propagator into account. The inverse propagator (4.6) corresponds to the particular case $f^{D_T} = f^{DD} = 1$, $f^{nD} = 0$, and $f^{nn} = 1/\xi$.

4.2 Expansion in the fields

Even for analytic calculations one wishes to include more than covariantly constant gauge fields, and to expand in powers of the fields, or to make a derivative expansion. Eventually one has to employ numerical methods where it is inevitable to make some sort of approximation. Therefore it is of importance to have a formulation of the flow equation which allows for simple and systematic expansions.

In this section we are arguing in favour for a different splitting of the propagator which makes it simple to employ any sort of approximation one may think of. For this purpose we employ the regulator $R_k[D^2(\bar{A})]$. This is an appropriate choice since it has no negative eigenvalues. We split the inverse propagator into

$$\Gamma^{(2)ab}_{k,\mu\nu}[A] = \Delta^{ab}_{\mu\nu} - \left(2gF^{ab}_{\mu\nu} - (D_\mu D_\nu)^{ab}\right) \tag{4.13}$$

with

$$\Delta^{ab}_{\mu\nu} = \left\{-D^2(1+r_D)\right\}^{ab}\delta_{\mu\nu} + \frac{1}{\xi n^2} n_\mu n_\nu \delta^{ab}\,. \tag{4.14}$$

The operator Δ can be explicitly inverted for any field configuration (and $A = \bar{A}$). We have

$$\Delta^{-1} = -\frac{1}{D^2(1+r_D)}\mathbb{1} + \frac{1}{D^2(1+r_D)}\frac{1}{1+\xi D^2(1+r_D)}P_n\,. \tag{4.15}$$

With (4.13) and (4.15) we can expand the propagator as

$$G_k[A,A] = \Delta^{-1}\sum_{n=0}^{\infty}\left[(2gF - D\otimes D)\,\Delta^{-1}\right]^n. \tag{4.16}$$

where $(D\otimes D)^{ab}_{\mu\nu}(x,y) = D^{ac}_\mu D^{cb}_\nu \delta(x-y)$. For $\xi = 0$ (the axial gauge), Δ^{-1} can be neatly written as

$$\Delta^{-1}(\xi=0) = -\frac{1}{D^2(1+r_D)}(\mathbb{1} - P_n)\,, \tag{4.17}$$

which simplifies the expansion (4.16). The most important points in (4.16) concern the fact that it is valid for arbitrary gauge field configurations and each term is convergent for arbitrary gauge fixing parameter ξ. Moreover such an expansion is not possible in the case of covariant gauges. Both facts mentioned above are spoiled in this case.

4.3 Heat kernels

We present closed formulae for the heat-kernel of the closely related operators D_T and $-D^2 = D_T + 2gF$. These are needed in order to evaluate the traces in (5.15). We define the heat-kernels as $K_{\mathcal{O}}(\tau) = \exp\{\tau\mathcal{O}\}(x,x)$

$$K_{D^2}(\tau) = \int \frac{d^4p}{(2\pi)^4} e^{\tau X_\mu X_\mu}\,, \tag{4.18a}$$

$$K_{-D_T}(\tau) = e^{2\tau F} K_{D^2}(\tau)\,, \tag{4.18b}$$

where $X_\mu = ip_\mu + D_\mu$ in the corresponding representation. Here we used that $2gF$ commutes with X_μ for covariantly constant fields. All kernels are tensors in the Lie algebra (K_{-D_T}

is also a Lorentz tensor because of the prefactor). For the calculation of the momentum integral we just refer the reader to the literature (e.g. [30]) and quote the result for covariantly constant field strength

$$K_{D^2}(\tau) = \frac{1}{16\pi^2\tau^2}\det\left[\frac{\tau g F}{\sinh\tau g F}\right]^{1/2}, \tag{4.19a}$$

$$K_{-D_T}(\tau) = \exp(2\tau g F)\, K_{D^2}(\tau)\,. \tag{4.19b}$$

Here, the determinant is performed only with respect to the Lorentz indices. For the computation of the one-loop beta function we need to know $K(\tau)$ in (4.19) up to order F^2 (equivalently to order τ^0). Expanding K_{D^2} in $\tau g F$ we get

$$K_{D^2}(\tau) = \frac{1}{16\pi^2}\left(\frac{1}{\tau^2} - \frac{1}{12}g^2(F^2)_{\rho\rho}\right) + O[\tau,(gF)^3]\,. \tag{4.20}$$

With (4.20) and the expansion $(\exp 2\tau g F)_{\mu\nu} = 1 + 2\tau g F_{\mu\nu} + 2\tau^2 g^2 (F^2)_{\mu\nu} + O[\tau,(gF)^3]$ we read off the coefficient of the $K(\tau)$ proportional to F^2,

$$\mathrm{Tr}\, K_{D^2}|_{F^2} = -\frac{1}{16\pi^2}\frac{4}{3}Ng^2\, S_A[A]\,, \tag{4.21a}$$

$$\mathrm{Tr}\, K_{-D_T}|_{F^2} = \frac{1}{16\pi^2}\frac{20}{3}Ng^2\, S_A[A]\,, \tag{4.21b}$$

where the trace Tr denotes a sum over momenta and indices. We have also used that $S_A[A] = \frac{1}{2}\int \mathrm{tr_f} F^2$ with $\mathrm{tr}_f\, t^a t^b = -\frac{1}{2}\delta^{ab}$. Since the operators D_T and D^2 carry the adjoint representation the trace Tr includes tr_{ad} with $2N\mathrm{tr_f} t^a t^b = \mathrm{tr_{ad}} t^a t^b$.

5. Applications

In order to put the methods to work we consider in this section the full one-loop effective action for SU(N) Yang-Mills theory which entails the universal one-loop beta function for arbitrary regulator function.

5.1 Effective action

For the right hand side of the flow we need

$$\Gamma_k[A,\bar A] = \frac{1}{2}\int Z_F(t)\,\mathrm{tr}_f\, F^2(A) + S_{gf}[A] + O[(gA)^5, g^2\partial A]\,, \qquad \mathrm{tr}_f\, t^a t^b = -\frac{1}{2}\delta^{ab} \tag{5.1}$$

where tr_R denotes the trace in the representation R, $R = f$ stands for the fundamental representation, $R = ad$ for the adjoint representation. Only the classical action can contribute to the flow, as n-loop terms in (5.1) lead to $n+1$-loop terms in the flow, when inserted on the right hand side of (3.9). This Ansatz leads to the propagator (4.11) which together with our choice for the regulator (3.19) is the input in the flow equation (3.9). We also use the following in the evaluation of the different terms in (3.9):

$$\mathrm{tr}\, D^2 = 4\mathrm{tr}\, D\otimes D \tag{5.2}$$

With this we finally arrive at

$$\partial_t \hat{\Gamma}_k = \frac{1}{2}\mathrm{Tr}\left\{\frac{\partial_t r(D_T)}{1+r(D_T)} - \frac{1}{2}\frac{\partial_t r(-D^2)}{1+r(-D^2)} + \frac{1}{4}\frac{\partial_t r(-D^2)}{s^2+r(-D^2)}\right\}, \tag{5.3}$$

where the trace Tr contains also the Lorentz trace and the adjoint trace tr_{ad} in the Lie algebra. The first term on the right-hand side in (5.3) has a non-trivial Lorentz structure, while the two last terms are proportional to $\delta_{\mu\nu}$. We notice that the flow equation (5.3) is well-defined in both the IR and the UV region. We apply the heat-kernel results of section 4.3 to the calculation of (5.3). To that end we take advantage of the following fact: Given the existence (convergence, no poles) of the Taylor expansion of a function $f(x)$ about $x=0$ we can use the representation

$$f(-\mathcal{O}) = f(-\partial_\tau)\exp\{\tau\mathcal{O}\}|_{\tau=0} \tag{5.4}$$

Due to the infrared regulator the terms in the flow equation (5.3) have this property, where $\mathcal{O} = D_T, D^2$. Hence we can rewrite the arguments D_T and $-D^2$ in (5.3) as derivatives w.r.t. τ of the corresponding heat kernels $K_{-D_T}(\tau)$ and $K_{D^2}(\tau)$. Applying this to the flow equation (5.3) we arrive at

$$\begin{aligned}\partial_t \hat{\Gamma}_k = \frac{1}{2}\Bigg[&\frac{\partial_t r(-\partial_\tau)}{1+r(-\partial_\tau)}\,\mathrm{Tr}K_{-D_T}(\tau) - \frac{1}{2}\frac{\partial_t r(-\partial_\tau)}{1+r(-\partial_\tau)}\,\mathrm{Tr}K_{D^2}(\tau) + \\ &+\frac{1}{4}\int dp_n \frac{(p_n^2-\partial_\tau)\partial_t r(p_n^2-\partial_\tau)}{p_n^2+(p_n^2-\partial_\tau)r(p_n^2-\partial_\tau)}\frac{\tau^{1/2}}{\sqrt{\pi}}\mathrm{Tr}K_{D^2}(\tau)\Bigg]_{\tau=0}\end{aligned} \tag{5.5}$$

The two terms in the first line follow from (5.3). The last term is more involved because it depends on both D^2 and nD due to $s^2 \equiv (nD)^2/n^2D^2$. We note that $nD = (n\partial)$ holds for configurations satisfying (4.1a) and only depends on the momentum parallel to n_μ. Furthermore it is independent of the gauge field. Now we use the splitting of $(p_\mu) = (p_n, \vec{p})$ where $p_n = P_n p$ and $\vec{p} = (1-P_n)p$. The heat kernel related to $\vec{D}^2$ follows from the one for D^2 via the relation $K_{\vec{D}^2}(\tau) = \frac{\tau^{1/2}}{\sqrt{\pi}}K_{D^2}(\tau)$ as can be verified by a simple gaussian integral in the p_n-direction.

With these prerequisites at hand, we turn to the full effective action at the scale k, which is given by

$$\hat{\Gamma}_k = \hat{\Gamma}_\Lambda + \int_\Lambda^k dk' \,\frac{\partial\hat{\Gamma}_{k'}}{\partial k'}\,, \tag{5.6}$$

where Λ is some large initial UV scale. We start with the classical action $\Gamma_\Lambda = S_A + S_{\mathrm{gf}}$. Performing the k-integral in (5.6) we finally arrive at

$$\begin{aligned}\hat{\Gamma}_k[A] = &\left(1+\frac{Ng^2}{16\pi^2}\left(\frac{22}{3}-7(1-\gamma)\right)\ln\frac{k}{\Lambda}\right)S_A[A] + \\ &+ S_{\mathrm{gf}}[A] + \sum_{m=1}^{\infty} C_m\left(\frac{k^2}{\Lambda^2}\right)\Delta\Gamma^{(m)}\left[\frac{gF}{k^2}\right] + \mathrm{const}\,.\end{aligned} \tag{5.7}$$

The combination $S_A + S_{\mathrm{gf}}$ on the right-hand side of (5.7) is the initial effective action. All further terms stem from the expansion of the heat kernels (4.19) in powers of τ. The terms $\sim \tau^{-2}$ give field-independent contributions, while those $\sim \tau^{-1}$ are proportional to $\mathrm{tr}F$ and vanish. The third term on the right-hand side of (5.7) stems from the τ^0 coefficient of the heat kernel. This term also depends on the regulator function through the coefficient γ (3.22). All higher order terms $\sim \tau^m, m > 0$ are proportional to the terms $C_m(k^2/\Lambda^2)\,\Delta\Gamma^{(m)}[gF/k^2]$. These terms have the following structure: They consists of a prefactor

$$C_m(x) = -\frac{1}{4m}\frac{(-)^m}{m!}(1-x^m) \tag{5.8a}$$

and scheme-dependent functions of the field strength, $\Delta\,\Gamma^{(m)}[gF]$, each of which is of the order $2+m$ in the field strength gF. They are given explicitly as

$$\Delta\Gamma^{(m)}[gF] = B_m^{D_T}\,\mathrm{Tr}\,K^{(m)}_{-D_T}(0) + \left(B_m^{D^2} + B_m^{nD}\right)\mathrm{Tr}\,K^{(m)}_{D^2}(0)\,. \tag{5.8b}$$

Here, $K^{(m)}_{D^2}(0)$ and $K^{(m)}_{-D_T}(0)$ denote the expansion coefficients of the heat kernels. We use the following identity

$$f^{(m)}(0) = f(\partial_\tau)\tau^m|_{\tau=0}\,, \tag{5.9}$$

and $f^{(m)}(x) = (\partial_x)^m f(x)$. In addition, the terms in (5.8b) contain the scheme-dependent coefficients

$$B_m^{D_T} = \left(\frac{\dot r_1}{1+r_1}\right)^{(m)}(0)\,, \tag{5.10a}$$

$$B_m^{D^2} = -\frac{1}{2}B_m^{D_T}\,, \tag{5.10b}$$

$$B_m^{nD} = \frac{(-1)^{m+1}}{4}\int_0^\infty dx\left(\partial_x - \frac{1}{x}\alpha\partial_\alpha\right)^{m+1}\frac{\dot r_1(x)}{\sqrt{r_1(x)}\sqrt{r_1(x)+\alpha}}\Bigg|_{\alpha=1}\,. \tag{5.10c}$$

The coefficients B^{D_T}, B^{D^2} and B^{nD} follow from the first, second and third term in (5.3). We introduced dimensionless variables by defining $r_1(x) = r(xk^2)$ and $\dot r_1(x) \equiv \partial_t r_1(x) = -2xk^2 r'(xk^2) = -2xr_1'(x)$, in order to simplify the expressions and to explicitly extract the k-dependence into (5.8a). The explicit derivation of B^{nD} is tedious but straightforward and is given — together with some identities useful for the evaluation of the integral and the derivatives — in appendix A. All coefficients B^{D_T}, B^{D^2} and B^{nD} are finite. The appearance of roots in the coefficient B^{nD} is not surprising after the discussion of the absence of spurious singularities in section 2.2.

In particular, we can read off the coefficients for $m=0$ which add up to the prefactor of the classical action in (5.7):

$$B_0^{D_T} = 2\gamma\,,\qquad B_0^{D^2} = -\gamma\,,\qquad B_0^{nD} = -\frac{1}{2}(1-\gamma)\,, \tag{5.11}$$

where we have used (A.5) in the appendix. Together with the heat kernel terms proportional to τ^0 given in (4.21) this leads to (5.7).

This application can be extended to include non-perturbative truncations. The flow of the coefficients (5.8b) becomes non-trivial, and regulator-dependent due to the regulator-dependence of the coefficients (5.10). Then, optimisation conditions for the flow can be employed to improve the truncation at hand [31].

Finally, we discuss the result (5.7) in the light of the derivative expansion. Typically, the operators generated along the flow have the structure $F\, f_k[(D^2+k^2)/\Lambda^2]\, F$, and similar to higher order in the field strength. For dimensional reasons, the coefficient function $f_k(x)$ of the operator quadratic in F develops a logarithm $\sim \ln x$ in the infrared region. An additional expansion of this term in powers of momenta leads to the spurious logarithmic infrared singularity as seen in (5.7). To higher order in the field strength, the coefficient function behave as powers of $1/(D^2+k^2)$, which also, at vanishing momenta, develop a spurious singularity in the IR, and for the very same reason. All these problems are absent for any finite external gluon momenta, and are an artifact of the derivative expansion. A second comment concerns the close similarity of (5.7) with one-loop expressions found within the heat-kernel regularisation. In the latter cases, results are given as functions of the proper-time parameter τ and a remaining integration over $d\ln\tau$. Expanding the integrand in powers of the field strength and performing the final integration leads to a structure as in (5.7), after identifying $\tau \sim k^{-2}$. In particular, these results have the same IR structure as found in the present analysis.

5.2 Running coupling

We now turn to the computation of the beta function at one loop. We prove that the result is independent of the choice of the regulator and agrees with the standard one. However, it turns out that the actual computation depends strongly on the precise small-momentum behaviour of the regulator, which makes a detailed discussion necessary.

Naively we would read-off the β-function from the t-running of the term proportional to the classical action S_A in (5.7). Using (3.12) leads to $\partial_t \ln Z_g = -\frac{1}{2}\partial_t \ln Z_F$. We get from (5.7)

$$Z_F = \left(\frac{22}{3} - 7(1-\gamma)\right)\frac{Ng^2}{16\pi^2}t \quad \rightarrow \quad \partial_t \ln Z_g = -\left(\frac{11}{3} - \frac{7}{2}(1-\gamma)\right)\frac{Ng^2}{16\pi^2} + O(g^4)\,. \tag{5.12}$$

We would like to identify $\beta = \partial_t \ln Z_g$. This relation, however, is based on the assumption that at one loop one can trade the IR scaling encoded in the t-dependence of this term directly to a renormalisation group scaling. This assumption is based on the observation that the coefficient of $S_A[A]$ is dimensionless and at one loop there is no implicit scale dependence. It is the latter assumption which in general is not valid. A more detailed analysis of this fact is given in [21]. Here, we observe that the background field dependence of the cut-off term inflicts contributions to $\partial_t Z_F S_{\rm cl}$. These terms would be regulator-dependent constants for a standard regulator without $\bar A$. As mentioned below (2.5), one should see the background field as an index for a family of different regulators. We write the effective action as

$$\Gamma_k[A,\bar A] = \Gamma_{k,1}[A] + \Gamma_{k,2}[\bar A] + \Gamma_{k,3}[A,\bar A]\,. \tag{5.13}$$

The second term only depends on $\bar A$ and is solely related to the $\bar A$-dependence of the regulator. The last term accounts for gauge invariance of Γ_k under the combined transformation $\boldsymbol{\delta}_\omega + \bar{\boldsymbol{\delta}}_\omega$. This term vanishes in the present approximation, because of the observation that our Ansatz is invariant — up to the gauge fixing term — under both $\boldsymbol{\delta}_\omega$ and $\bar{\boldsymbol{\delta}}_\omega$ separately. The physical running of the coupling is contained in the flow of $\Gamma_{k,1}[A]$. This leads to

$$\beta = -\frac{1}{2}\partial_t Z_F + \frac{1}{2}\partial_t Z_{F,2}, \tag{5.14}$$

where $Z_{F,2}$ is the scale dependence of $\Gamma_{k,2} \propto Z_{F,2} S_A[A]$. We rush to add that this procedure is only necessary because we are interested in extracting the universal one-loop β-function from the flow equation. For integrating the flow itself this is not necessary since for $k = 0$ the background field dependence disappears anyway. For calculating $\partial_t \ln Z_{F,2}$ we use (4.11) and (5.2) and get

$$\begin{aligned}\partial_t \frac{\delta}{\delta \bar{A}^a_\mu}\Gamma_k[A, \bar{A} = A] = \frac{1}{2}\mathrm{Tr}\partial_t \Bigg\{ & \frac{R'_k[D_T]}{D_T + R_k[D_T]}\frac{\delta D_T}{\delta \bar{A}^a_\mu} + \frac{1}{2}\frac{R'_k(-D^2)}{-D^2 + R_k[-D^2]}\frac{\delta D^2}{\delta \bar{A}^a_\mu} - \\ & - \frac{1}{4}\frac{R'_k[-D^2]}{(-nD)^2 + R_k[-D^2]}\frac{\delta D^2}{\delta \bar{A}^a_\mu}\Bigg\},\end{aligned} \tag{5.15}$$

where we have introduced the abbreviation

$$R'_k(x) = \partial_x R_k(x)\,. \tag{5.16}$$

For the derivation of (5.15) one uses the cyclycity of the trace and the relations (4.2). We notice that (5.15) is well-defined in both the IR and the UV region. The explicit calculation is done in appendix B. Collecting the results (B.2),(B.3),(B.4) we get

$$\partial_t \delta_{\bar{A}}\Gamma_k[A, \bar{A} = A]|_{F^2} = -\frac{Ng^2}{16\pi^2}\,7(1-\gamma)\,\delta_A S_A[A] \quad \to \quad \partial_t Z_{F,2} = -\frac{Ng^2}{16\pi^2}\,7(1-\gamma)\,. \tag{5.17}$$

We insert the results (5.12) for $\partial_t Z_F$ and (5.17) for $\partial_t Z_{F,2}$ in (5.14) and conclude

$$\beta = -\frac{11}{3}\frac{Ng^2}{16\pi^2} + O(g^4)\,. \tag{5.18}$$

which is the well-known one-loop result. For regulators with a mass-like infrared limit, $\gamma = 1$, there is no implicit scale dependence at one loop. It is also worth emphasising an important difference to Lorentz-type gauges within the background field approach. In the present case only the *physical* degrees of freedom scale implicitly with $t = \ln k$ for $\gamma \neq 0$. This can be deduced from the prefactor $7(1-\gamma)$ in (5.17). Within the Lorentz-type background gauge, this coefficient is $\frac{22}{3}(1-\gamma)$ [21]. The difference has to do with the fact that in the axial gauge one has no auxiliary fields but only the physical degrees of freedom. In a general gauge, this picture only holds true after integrating-out the ghosts. This integration leads to non-local terms. They are mirrored here in the non-local third term on the right hand side of the flow (5.5) and in the third term on the right hand side of (5.15) (see also (B.4)).

6. Conclusions

We have shown how the exact renormalisation group can be used for gauge theories in general axial gauges. We have addressed various conceptual points, in particular the absence of spurious singularities and gauge invariance, which are at the basis for a reliable application of this approach. We have shown that spurious singularities are absent provided that the regulator R_k decays stronger than $(p^2)^{-4}$ for large momenta. In turn, regulators with milder decay are highly questionable. At least they are subject to a renormalisation of the flow itself, which implicitly brings back the problem of spurious singularities. This concerns in particular the mass regulator $R_k = k^2$, see also [3].

Our main goal was to develop methods which allow controlled and systematic analytical considerations. The formalism has the advantage that ghost fields are not required. Also, no additional regularisation — in spite of the axial gauge fixing — is needed. This is a positive side effect of the wilsonian regulator term. In addition, we worked in a background field formulation, which is helpful in order to construct a gauge invariant effective action. Also, it allows to expand the flow equation around relevant field configurations. Instead of relying on the standard background field gauge, we have introduced the background field only in the regulator term. The axial gauge fixing is independent on the background field. This way, it is guaranteed that the background field dependence vanishes in the IR limit. It is important to discuss how this differs from the usual background field approach to wilsonian flows. In both cases, applications of the flow require an approximation, where derivatives w.r.t. the background field are neglected, cf. (3.18). In the present approach, this approximation improves in the infrared, finally becoming exact for $k = 0$ as the background field dependence disappears. For the background field gauge this does not happen, because the full effective action still depends non-trivially on the background field.

As an application, the full one-loop effective action and the universal beta-function have been computed. This enabled us to address some of the more subtle issues of the formalism like the implicit scale dependence introduced by the cutoff, which has properly to be taken into account for the computation of universal quantities, and the scheme independence of the beta-function. The equation which controls the additional background field dependence introduced by the cutoff contains the related information.

These results are an important step towards more sophisticated applications, both numerically and analytically. A natural extension concerns dynamical fermions. The present formalism is also well-adapted for QCD at finite temperature T, where the heat-bath singles-out a particular Lorentz vector. Here, an interesting application concerns the thermal pressure of QCD.

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A. Evaluation of the one loop effective action

The calculation of the last term in (5.7) is a bit more involved. Note that the following argument is valid for $m \geq -1$, $m > -1$ is of importance for the evaluation of (5.7), $m = -1$ will be used in appendix B. We first convert the factor $\tau^{m+1/2}$ appearing in the expansion of the heat kernel using $\tau^{1/2+m} = (-1)^{m+1}\frac{\tau}{\sqrt{\pi}}\int dz\partial_{z^2}^{m+1}e^{-\tau z^2}$. We further conclude that

$$\begin{aligned} B_m^{nD} &= \frac{1}{4\pi}\int dp_n\, dz\, \frac{(p_n^2-\partial_\tau)\partial_t r(p_n^2-\partial_\tau)}{p_n^2+(p_n^2-\partial_\tau)r(p_n^2-\partial_\tau)}\tau^{m+1}e^{-\tau z^2}\Big|_{\tau=0} \\ &= \frac{(-1)^{m+1}}{4\pi}\int dp_n\, dz\, \partial_{z^2}^{m+1}\frac{\partial_t r(p_n^2-\partial_\tau)}{p_n^2+(p_n^2-\partial_\tau)r(p_n^2-\partial_\tau)}(p_n^2-\partial_\tau)e^{-\tau z^2}\Big|_{\tau=0} \\ &= \frac{(-1)^{m+1}}{4\pi}\int dp_n dz \partial_{z^2}^{m+1}\frac{\partial_t r(z^2+p_n^2)}{\frac{p_n^2}{z^2+p_n^2}+r(z^2+p_n^2)}\,, \end{aligned} \tag{A.1}$$

The expression in (A.1) can be conveniently rewritten as

$$\begin{aligned} B_m^{nD} &= \frac{(-1)^{m+1}}{8\pi}\int_0^\infty dx\int_0^{2\pi}d\phi\left(\partial_x-\frac{1}{x}\alpha\partial_\alpha\right)^{m+1}\frac{\partial_t r(x)}{\alpha\sin^2\phi+r(x)}\Big|_{\alpha=1} \\ &= \frac{(-1)^{m+1}}{4}\int_0^\infty dx\left(\partial_x-\frac{1}{x}\alpha\partial_\alpha\right)^{m+1}\frac{\partial_t r(x)}{\sqrt{r(x)}\sqrt{r(x)+\alpha}}\Big|_{\alpha=1}\,. \end{aligned} \tag{A.2}$$

where $x = z^2+p_n^2$ and $\sin^2\phi = p_n^2/(z^2+p_n^2)$. It is simple to see that $-(1/x)\alpha\partial_\alpha$ is a representation of ∂_{z^2} on $\sin^2\phi = p_n^2/(z^2+p_n^2)$ and ∂_x a representation of ∂_{z^2} on functions of x only. The expression in (A.2) is finite for all $m \geq 0$. Evidently it falls of for $x \to \infty$. For the behaviour at $x = 0$ the following identity is helpful:

$$\left(\partial_x-\frac{1}{x}\alpha\partial_\alpha\right)^{m+1} = \sum_{i=0}^{m+1}(-1)^{m+1-i}\binom{m+1}{i}\partial_x^i\left(\frac{\alpha}{x}\right)^{m+1-i}\partial_\alpha^{m+1-i}\,, \tag{A.3}$$

Eq. (A.3) guarantees that the integrand in (A.2) only contains terms of the form

$$\partial_x^i\left(\frac{\dot r}{\sqrt{r}\sqrt{1+r}}(x+xr)^{i-m-1}\right) \tag{A.4}$$

with $i = 0, \ldots, m+1$. For $x \to 0$ one has to use that $\partial_t r \to 2nr$ and $r \to k^{2n}/x^n$. The terms of integrand in (A.2) as displayed in (A.4) are finite for $x = 0$.

We are particularly interested in B_0^{nD} relevant for the coefficient of S_A in the one loop effective action (5.7). With (A.2) it follows

$$\begin{aligned} B_0^{nD} &= -\frac{1}{4}\int_0^\infty dx\left(\partial_x-\frac{1}{x}\alpha\partial_\alpha\right)\frac{\partial_t r(x)}{\sqrt{r(x)}\sqrt{r(x)+\alpha}}\Big|_{\alpha=1} \\ &= -\frac{1}{4}\left(\frac{\partial_t r(x)}{\sqrt{r(x)}\sqrt{1+r(x)}}-2\frac{\sqrt{r(x)}}{\sqrt{1+r(x)}}\right)_{x=0}^{x=\infty} = -\frac{1}{2}(1-\gamma)\,, \end{aligned} \tag{A.5}$$

where we have used $\partial_t r(z) = -2z\partial_z r(z)$ and the limits for $\partial_t r(z\to 0) = 2\gamma z^{-\gamma}$, $r(z\to 0) = z^{-\gamma}$, $r(z\to\infty) = 0$.

B. $\bar{A}$-derivatives

For the calculation of (5.15) the following identity is useful:

$$\mathrm{Tr}\left(\frac{\delta}{\delta A_\mu^a}\mathcal{O}\right)e^{\tau\mathcal{O}} = \frac{1}{\tau}\mathrm{Tr}\frac{\delta}{\delta A_\mu^a}e^{\tau\mathcal{O}}, \tag{B.1}$$

where we need (B.1) for $\mathcal{O} = D^2$ and $\mathcal{O} = -D_T$. Now we proceed in calculating the first term in (5.15) by using a similar line of arguments as in the calculation of (5.7) and in appendix A. We make use of the representation of $\tau^{-1} = \int_0^\infty dz \exp -\tau z$ and arrive at

$$\begin{aligned}\frac{1}{2}\mathrm{Tr}\,\partial_t\left(\frac{R_k'[D_T]}{D_T + R_k[D_T]}\frac{\delta D_T}{\delta A_\mu^a}\right) &= \frac{1}{2}\mathrm{Tr}\,\partial_t\left(\frac{R_k'(-\partial_\tau)}{-\partial_\tau + R_k[-\partial_\tau]}\frac{1}{\tau}\frac{\delta}{\delta A_\mu^a}K_{-D_T}(\tau)\right)_{\tau=0}\\ &= \frac{1}{2}\int_0^\infty \frac{dx}{x}\partial_t\left(\frac{R_k'[x]}{1+r[x]}\right)\frac{Ng^2}{16\pi^2}\frac{20}{3}\frac{\delta}{\delta A_\mu^a}\left(S_A[A] + O[g]\right)\\ &= -\frac{Ng^2}{16\pi^2}\frac{20}{3}(1-\gamma)\frac{\delta}{\delta A_\mu^a}\left(S_A[A] + O[g]\right).\end{aligned} \tag{B.2}$$

Note that ∂_t acts as $-2x\partial_x$ on functions which solely depend on x/k^2. The term $R'/(1+r)$ is such a function. The second term can be calculated in the same way leading to

$$\begin{aligned}\frac{1}{4}\mathrm{Tr}\,\partial_t\left\{\frac{-R_k'[D^2]}{-D^2 + R_k[-D^2]}\frac{\delta}{\delta A_\mu^a}D^2\right\} &= \frac{1}{4}\int_0^\infty \frac{dx}{x}\,\partial_t\left(\frac{R_k'[x]}{1+r[x]}\right)\frac{Ng^2}{16\pi^2}\frac{4}{3}\frac{\delta}{\delta A_\mu^a}\left(S_A[A] + O[g]\right)\\ &= -\frac{Ng^2}{16\pi^2}\frac{2}{3}(1-\gamma)\frac{\delta}{\delta A_\mu^a}\left(S_A[A] + O[g]\right).\end{aligned} \tag{B.3}$$

The calculation of the last term in (5.15) is a bit more involved, but boils down to the same structure as for the other terms. Along the lines of appendix A it follows that this term can be written as

$$\begin{aligned}\frac{1}{8}\mathrm{Tr}\,\partial_t\left\{\frac{-R_k'[-D^2]}{(-nD)^2 + R_k[-D^2]}\frac{\delta D^2}{\delta A_\mu^a}\right\} &= \frac{1}{8}\mathrm{Tr}\,\partial_t\left\{\int dp_n\frac{R_k'[p_n^2-\partial_\tau]}{p_n^2+R_k[p_n^2-\partial_\tau]}\frac{\tau^{-1/2}}{\sqrt{\pi}}\frac{\delta}{\delta A_\mu^a}K_{D^2}(\tau)\right\}_{\tau=0}\\ &= -\frac{1}{8}\int_0^\infty\frac{dx}{x}\partial_t\frac{R_k'}{\sqrt{r}\sqrt{1+r}}\frac{Ng^2}{16\pi^2}\frac{4}{3}\frac{\delta}{\delta A_\mu^a}\left(S_A[A] + O[g]\right),\\ &= \frac{Ng^2}{16\pi^2}\frac{1}{3}(1-\gamma)\frac{\delta}{\delta A_\mu^a}\left(S_A[A] + O[g]\right)\end{aligned} \tag{B.4}$$

Note that when rewriting the left hand side of (B.4) as a total derivative w.r.t. A this also includes a term which stems from $\frac{\delta}{\delta A}(nD)^2$. This, however, vanishes because it is odd in p_n.

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